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Mixed H_2/H_∞ Control of Continuous-time Singularly Perturbed System–state Feedback Computations

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This study brings out the scheme for the design of a mixed H_2/H_∞ based feedback controller for a continuous-time singularly perturbed system using state feedback computations. The mixed H_2/H_∞ control law was derived using the auxiliary cost minimisation approach and the feedback controller was formulated for a linear time invariant lower and higher order continuous-time singularly perturbed systems by solving iteratively coupled Riccati equations. The H_∞-controller based on the mixed sensitivity approach and the Linear Quadratic Gaussian (LQG) controller were derived for the same system. The time responses for unit step input and robustness properties such as Gain and Phase margin were studied by formulating mixed H_2/H_∞, H_∞ and LQG systems.

Key words: *H_2/H_∞ control, singularly perturbed system, H_∞ and LQG control, state feedback, Riccati equation, continuous-time*

1.0 INTRODUCTION

The linear state feedback regulators are designed for singularly perturbed systems based on Linear Quadratic Regulation (LQR) by minimising quadratic performance index as discussed in the literature [1]-[3]. These regulators could not guarantee good performance against measurement noise disturbances acting on the plant. The pioneering work in H_∞ optimisation was initiated by G Zames [4], where the disturbances of partially unknown dynamics and the noise information could be modelled and the concept was used to obtain the optimal feedback controller, which minimises the H_∞-norm of the weighted sensitivity function, while achieving the internal stability. The control theory studied in [5]-[8], provides the controller which is robust against uncertainties and worst-case disturbances normally encountered in the plants, its major disadvantage being that the order of the controller is many times higher than the plant's order. In spite of the fundamental difference between LQG design and H_∞ theory, a significant connection was discovered in [9], in which it was observed that a modified algebraic Riccati equation developed for parameter robust full state feedback control could yield controllers satisfying H_∞ disturbance attenuation bounds. This relation was further explored in [10], where the state-space approach developed solved the problem of finding among all state feedback controllers that minimise an H_2 performance measure, one that also satisfies an H_∞-norm bound.

The standard H_2 and H_∞ optimal control problems are treated as separate problems, but in a unified state-space framework as discussed in [11]. This motivated the later researchers to continue

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this work to find a single problem formulation that has the standard H_2 and H_∞ theories as special cases. The mixed H_2/H_∞ problem was also studied in the form of Constrained Optimal Control Problem (COCP), where an admissible controller (K) was determined for the plant (G), which minimises the 2-norm of the transfer function [Tz_0w_0, where w_0 is input, z_0 is output] while satisfying the H_∞-constraint [$\|Tz_1w_1\|_\infty < \gamma$, where w_1 is input, z_1 is output] for a given $\gamma > 0$. Among all the admissible controllers, K provides robust stability, the COCP is to find a controller that minimises the variance of output z_0 (with $\Delta = 0$), when the input w_0 is zero mean unit variance white noise. However, there is no analytical solution to this problem, but some attempts have been made to solve modified versions of the optimisation problem by Mustafa and Glover [12].

In the mixed H_2/H_∞ control, the quadratic performance and disturbance attenuation criteria are tied in an auxiliary minimisation problem, which provides a controller of fixed order and achieves worst-case disturbance attenuation but sacrificing quadratic performance to some extent. The Bernstein-Haddad [13] has formulated auxiliary cost mixed H_2/H_∞ problem, where the exogenous input vector and the nominal performance were measured with an upper bound to the 2-norm of the transfer matrix. The necessary and suffcient conditions for optimal controllers were given interms of coupled Riccati equations. This approach is exploited here for the design of the mixed H_2/H_∞ controller for singularly perturbed systems taking advantage of the reduced order model structure. These controllers can provide both robust stability (via closed-loop H_∞-norm bound) and nominal performance (via a closed-loop LQG cost bound or H_2-norm bound). In this study, the mixed H_2/H_∞ based state feedback controller was derived for continuous time SPS by solving coupled Riccati equations. The procedure developed drastically, reduces the computational complexities involved in solving the coupled Riccati equations. The mixed H_2/H_∞ control law thus derived has been applied to continuous-time singularly perturbed systems. The coupled Riccati equations are solved iteratively using continuation method [13]. The H_∞-controller and the LQG controller were also formulated for the same system. The time responses of the closed-loop mixed H_2/H_∞ LQG and H_∞ system for unit step input and robustness properties such as Gain and Phase Margin were analysed.

2.0 FORMULATION OF MIXED H_2/H_∞ CONTROL PROBLEM

The H_2/H_∞ control theory described in [13] is presented in this section for a continuous time system with a singularly perturbed structure for a linear state variable feedback (LSVF). The basic structure of the feedback system in the form of a block diagram is shown in Fig. 1, where 'G' is the generalised plant and K is the controller. The disturbance '*w*' is a zero mean white noise, the '*z*' is the output to be controlled; and '*y*' is the measured variable and '*u*' is the control input.

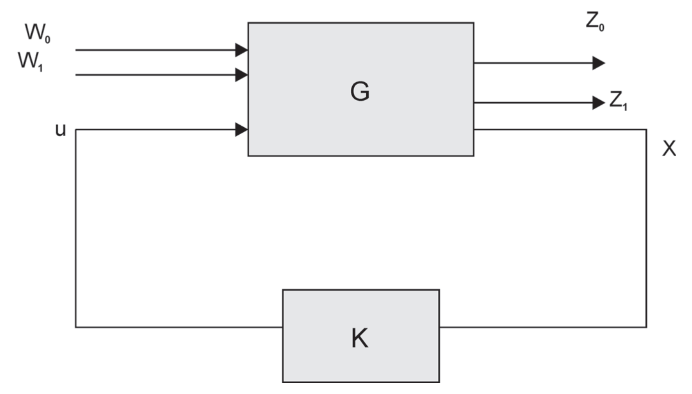

FIG. 1 LINEAR TIME INVARIANT FEEDBACK SYSTEM

The linear time invariant singularly perturbed system 'G' can be described by the equation:

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = Ax(t) + B_1 w(t) + B_2 u(t) \quad (1)$$

with $$\begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$$

The controlled output z*(t)* and the measured output *y(t)* respectively are:

$$z(t) = C_1 x(t) + D_1 u(t) \quad (2)$$

$$y(t) = C_2 x(t) + D_2 w(t) \quad (3)$$

where *w(t)* is the disturbance input with E[w(t)w' (t)] = I, (A,B_1) is stabilisable and (C_1,A) is detectable, where:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ \frac{A_{21}}{\varepsilon} & \frac{A_{22}}{\varepsilon} \end{bmatrix}; B_1 = \begin{bmatrix} B_{11} \\ \frac{B_{21}}{\varepsilon} \end{bmatrix}; B_2 = \begin{bmatrix} B_{12} \\ \frac{B_{22}}{\varepsilon} \end{bmatrix}; \; C_1 = [C_{11} \quad C_{12}]; C_2 = [C_{21} \quad C_{22}]$$

with a linear state variable feedback (LSVF) control law:

$$u = Kx = [K_1 \quad K_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (4)$$

(i) the closed-loop system (1)-(4), viz.,

$$\dot{x}(t) = \tilde{A}x(t) + \tilde{B}w(t) \quad (5)$$

$$z(t) = \tilde{C}x(t) \quad (6)$$

$\tilde{A}$ is asymptotically stable, where:

$$\tilde{A} = A + B_2 K = \begin{bmatrix} A_{11} & A_{12} \\ \frac{A_{21}}{\varepsilon} & \frac{A_{22}}{\varepsilon} \end{bmatrix} + \begin{bmatrix} B_{12} \\ \frac{B_{22}}{\varepsilon} \end{bmatrix} [K_1 \quad K_2] = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \frac{\tilde{A}_{21}}{\varepsilon} & \frac{\tilde{A}_{22}}{\varepsilon} \end{bmatrix}$$

$$\tilde{B} = B_1 = \begin{bmatrix} B_{11} \\ \frac{B_{21}}{\varepsilon} \end{bmatrix}; \; \tilde{C} = C_1 + D_1 K = [C_{11} + D_1 K_1 \quad C_{12} + D_1 K_2] = [\tilde{C}_{11} \quad \tilde{C}_{12}]$$

$$For \;\; i, \, j = 1, 2 \qquad \tilde{A}_{ij} = A_{ij} + B_{i2} K_j \; ; \; \tilde{C}_{1i} = C_{1i} + D_1 K_i$$

(ii) the transfer function from *w(t)* to *z(t)*,

$$T_{zw}(s) = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} \quad (7)$$

satisfies the H_∞ constraint:

$$\|T_{zw}\|_\infty \le \gamma \;\; (\, where \;\; \gamma \succ 0 \,) \quad (8)$$

and

(iii) the performance functional

$$J(K) = trace \lim_{t\to\infty} \mathrm{E}\{x'(t)R_1x(t) + 2x'(t)R_{12}u(t) + u'(t)R_2u(t) \quad (9)$$

is minimised, where E{.} stands for expected value of {.}

Now, with

$R_1 := S_1{'}S_1 \quad R_2 := S_2{'}S_2 \quad R_{12} := S_1{'}S_2$

and S_1 further partitioned as $S_1 = [S_{11} \quad S_{12}]$

Equation (9) is simplified using equation (4) [u = Kx(t)]

$$\begin{aligned} J(K) &= trace \lim_{t\to\infty} \mathrm{E}[x'(t)\{(S_1 + S_2K)'(S_1 + S_2K)\}x(t)] \\ &= tr\, \tilde{Q}\tilde{R} \end{aligned} \quad (10)$$

where:

$$\begin{aligned} \tilde{R} &= (S_1 + S_2K)'(S_1 + S_2K) \\ \tilde{Q} &= \lim_{t\to\infty} \mathrm{E}[x(t)x'(t)] \end{aligned}$$

3.0 DESIGN OF MIXED H_2/H_∞ CONTROLLER

The aim of the mixed H_2/H_∞ problem is to minimise the H_2 performance index *J(K)* equation (10) subject to the H_∞-bound $\|Tzw\|_\infty \leq \gamma$ for the robust stability. There is no analytical solution to this problem and hence the original problem is relaxed in an auxiliary minimisation problem. In [11], an attempt is made to develop state feedback controllers that minimise the H_2-norm of a closed-loop transfer matrix and also obtain necessary and sufficient conditions for the existence of a controller that also satisfies a prescribed H_∞-norm. When these conditions are satisfied, the solution of the problem is also a global solution to the constrained optimisation problem of minimising an H_2-norm performance measure subject to an H_∞- constraint. Another suboptimal approach known as the auxiliary minimisation problem was introduced by Bernstein and Haddad in [13]. Here, to enforce the H_∞- constraint, an upper-bound for the H_2 criteria is derived. The minimisation of the upper bound shows that the enforcement of an H-infinity disturbance attenuation constraint leads directly to an increase in the H_2 performance bound. The present study follows the above suboptimal approach for formulating a LSVF controller for a singularly perturbed system. To enforce the disturbance attenuation constraint (8), the Lyapunov equation is replaced with an Algebraic Riccati Equation (ARE), the solution of which over-bounds the closed-loop steady state covariance, in view of the following theorem.

Theorem 1: Let $K = [K_1 \quad K_2]$ be given and assume that $\exists Q \in R^{n\times n}$, a non-negative definite matrix-satisfying:

$$\tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}' + \gamma^{-2}Q\tilde{C}'\tilde{C}Q + \tilde{B}\tilde{B}' = 0 \quad (11)$$

then,

(a) $(\tilde{A}, \tilde{B})$ is stabilisable iff $\tilde{A}$ is asymptotically stable. In this case

(b) $\|Tzw\|_\infty \leq \gamma$ and (c) $\tilde{Q} \leq Q$, consequently (d) J(K)≤J(K,Q), where J(K,Q)= trace[QR̃].

This theorem shows that if a positive semi-definite solution exists for Q, then

$$J(K,Q) = trace[Q\tilde{R}] \tag{12}$$

leads to an upper-bound on the H_2-performance criteria. The following theorem guarantees the existence of a unique nominal positive semi-definite solution to (equation 12), when the H-infinity constraint is satisfied.

Theorem 2 : Suppose that $K \in R^{ml\ x\ n}$ is given, which leads to an asymptotically stable $\tilde{A}$ and the disturbance attenuation constraint $\|Tzw\|_\infty \leq \gamma$ is satisfied. Then, there exists a unique non-negative definite solution for Q satisfying:

$$\tilde{A}Q + Q\tilde{A}' + \gamma^{-2}Q\tilde{C}'\tilde{C}Q + \tilde{B}\tilde{B}' = 0 \tag{13}$$

and such that $(\tilde{A} + \gamma^{-2}Q\tilde{C}'\tilde{C})$ is asymptotically stable and further the solution obtained is minimal.

3.1 Auxiliary minimisation of mixed H_2/H_∞ problem

The auxiliary minimisation problem is described for the state feedback case. For the given system, the objective is to determine the feedback gain K, which minimises the performance functional J(K,Q)= tr(QR̃) subject to the constraint $\|Tzw\|_\infty \leq \gamma$. The equation (13) is the algebraic Riccati equation for a dual system in which the roles of input and output matrices are interchanged if compared with the following continuous time Riccati equation:

$$0 = QA + A'Q - QLQ + C'C; \quad \text{where } L = BR^{-1}B'$$

So the structure of Q can be shown to be of the form given in [1]:

$$Q = \begin{bmatrix} Q_1 & Q_2 \\ Q'_2 & Q_3/\varepsilon \end{bmatrix} \tag{14}$$

which is different from what is given in H_2 optimisation problem. In view of Theorem1, if a positive semi-definite solution exists to equation (12), then equation (13) becomes an upper-bound to the H_2 performance functional and simultaneously the H-infinity disturbance attenuation constraint (8) is satisfied. Instead of directly minimising (10), the idea is to minimise an auxiliary cost (13) for a fixed controller structure. In this way the original H_2/H_∞ design criterion is relaxed. Following the procedure laid down in [15], the auxiliary cost is minimised by forming the Lagrangian without any loss of generality:

$$L(K,Q,P) = tr\{Q\tilde{R} + (\tilde{A}Q + Q\tilde{A}' + Q\tilde{R}_\infty Q + \tilde{V})P\} \tag{15}$$

where $\tilde{R}_\infty = \gamma^{-2}\tilde{C}'\tilde{C}$; $\tilde{V} = \tilde{B}\tilde{B}'$ and 'P' is the positive semi-definite Lagrangian multiplier. The optimal conditions are: $\partial L / \partial Q = 0$ and $\partial L / \partial K = 0$ which respectively result as follows:

$$\frac{\partial L}{\partial Q} = 0 = \tilde{R}' + \tilde{A}'P + P\tilde{A} + (\tilde{R}_\infty QP)' + \tilde{R}_\infty QP$$
$$0 = (\tilde{A} + Q\tilde{R}_\infty)'P + P(\tilde{A} + Q\tilde{R}_\infty) + \tilde{R} \tag{16}$$

This is the full-order Lagrangian multiplier equation. Here $(\tilde{A} + Q\tilde{R}_\infty)$ is assumed to be stable.

3.2 Derivation of control law

In order to design the mixed H_2/H_∞ controller for a continuous time singularly perturbed system, the three coupled Riccati equations are solved iteratively by determining the feedback control gain matrix K. The full-order controller gain equation is obtained by substituting $\tilde{R}$, $\tilde{A}$ and $\tilde{C}$, we have:

$$L(K, Q, P) = tr\{Q(S_1'S_1 + Q(S_1'S_2K + QK'S_2'S_1 + QK'S_2'S_2K + AQP + B_2KQP + QA'P + \tilde{B}\tilde{B}'P$$
$$+ QK'B_2'P + \gamma^{-2}QC_1'C_1QP + \gamma^{-2}QC_1'D_1KQP + \gamma^{-2}QK'D_1'C_1QP + \gamma^{-2}QK'D_1'D_1KQP\}$$
$$\frac{\partial L}{\partial K} = 0 = S_2'S_1Q + S_2'S_1Q + S_2'S_2KQ + S_2'S_2KQ + B_2'PQ + B_2'PQ + \gamma^{-2}D_1'C_1QPQ$$
$$+ \gamma^{-2}D_1'C_1QPQ + \gamma^{-2}D_1'D_1KQPQ + \gamma^{-2}D_1'D_1KQPQ$$
$$0 = S_2'S_1Q + S_2'S_2KQ + B_2'PQ + \gamma^{-2}D_1'C_1QPQ + \gamma^{-2}D_1'D_1KQPQ$$
$$0 = [S_2'(S_1 + S_2K) + \{B_2' + \gamma^{-2}D_1'(C_1 + D_1K)Q\}P]Q \tag{17}$$

For the central controller,

$$S_1 = [S_{11} \quad S_{12}], \quad C_1 = [C_{11} \quad C_{12}], \quad S_2 = D_1$$
$$0 = D_1'((C_1 + D_1K)(I + \gamma^{-2}QP) + B'_2 P \tag{18}$$

This is the full-order gain equation. The coupled equations (13), (16) and (18) may be solved iteratively for Q, P and the desired control K in the full-order case. Here, the Q equation can be solved independently as it is free of P and K. The P equation, however, depends on Q and the gain equation depends on both Q and P. So, P and K equations are to be solved simultaneously.

On the other hand if we assume that $D_1' [C_1 \;\; D_1] = [0 \;\; I]$, then K may be singled out as:

$$K = -B'_2 P(I + \gamma^{-2}QP)^{-1} \tag{19}$$

By substituting above K, the Riccati and Lagrangian equations obtained are:

$$0 = AQ + QA' + \gamma^{-2}QC'_1 C_1Q + B_1B'_1 - \gamma^2 B_2B'_2 + \gamma^2 (I + \gamma^{-2}QP)^{-1} B_2B'_2 (I + \gamma^{-2}PQ)^{-1} \tag{20}$$

$$0 = PA + A'P + (I + \gamma^{-2}PQ)C'_1 C_1 + \gamma^{-2}C'_1 C_1QP - P(I + \gamma^{-2}PQ)^{-1} B_2B'_2 (I + \gamma^{-2}QP)^{-1} P \tag{21}$$

In the full-order case, we can solve simultaneously the coupled equations (20) and (21) for Q and P, as they are independent of K, and finally the full-order mixed H_2/H-infinity feedback gain K can be computed as it is a function of both Q and P.

4.0 RESPONSE AND ROBUSTNESS MEASURES OF MIXED H_2/H_∞, LQG AND H_∞ CONTROLLERS

The design of control systems almost invariably involves trade-offs among competing objectives. It is often the case that the controller is required to meet several different performance and robustness goals, and all of these cannot be met simultaneously. For example, it is intuitively clear that to obtain robust stability margin, it is likely that the performance criterion of the control system needs to be compromised. In this section the mixed H_2/H_∞, LQG and H_∞ controllers were formulated for the second and fourth order linear time invariant continuous time singularly perturbed systems described below.

$$\begin{bmatrix} \dot{x}(t) \\ \varepsilon\dot{z}(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + B_1 w(t) + B_2 u(t) \tag{22}$$

$$y(t) = C_{11}x(t) + C_{12}z(t) + D_1 u(t) \tag{23}$$

where $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, $z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$ *and* $C_1 = \begin{bmatrix} C_{11} & C_{12} \end{bmatrix}$

Description of systems

The second and fourth order systems considered are:

$$A = \begin{bmatrix} -2 & 1 \\ -10 & 20 \end{bmatrix},\ B_1' = \begin{bmatrix} 1 & 12 \end{bmatrix},\ B_2' = \begin{bmatrix} 1 & 1 \end{bmatrix},\ C_1 = \begin{bmatrix} 1.05 & 0.8 \end{bmatrix} \text{ and } D_1 = [1]$$

$$A = \begin{bmatrix} -0.2 & 0.2 & 0 & 0 \\ 0 & -0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -2 \end{bmatrix},\ B_1' = B_2' = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix},\ C_1 = \begin{bmatrix} 1 & 0.8 & 0.65 & 1 \end{bmatrix} \text{ and } D_1 = [0]$$

Where A is the system matrix, B_1 is the disturbance matrix, B_2 is the control matrix and C_1 is the measurement matrix.

4.1 Formulation of mixed H_2/H_∞ controllers

The control input is defined as u = K x(t), where K is a linear state variable feedback control that satisfies: (i) that the closed-loop system is asymptotically stable (ii) the closed-loop transfer function satisfies the constraint $\|T(s)\|_\infty \le \gamma$, where γ is a constant (iii) the performance functional $J(K)=\lim_{t\to\infty}\{[x(t)+u(t)]'[x(t)+u(t)]\}$ is to be minimised. To find out the solution to this problem, the coupled matrix Riccati equations (19), (20) and (21) obtained in the previous section using state feedback computations are solved sequentially. The idea is to exploit the fact that for large γ, the problem is approximated by LQG, which provides a reliable starting solution. The continuation parameter γ is then successively decreased until the desired value of γ is achieved, where Q and P will be positive definite. Using the Robust Control System tool box of MATLAB software, the equations for Q, P and K are solved iteratively to obtain the appropriate control law. The simplified sequence of continuation algorithm used is: (a) initially neglect γ and find the initial values of Q and P (b) set γ to some high value (c) determine the Q and P using equations (23)-(24) (d) check the positive definiteness of Q and P (e) if Q and P are positive definite, compute the feedback gain K (e) else, decrease γ and repeat steps (c)-(e). The value of constraint constants (γ) obtained for the second and third order systems respectively are 1.15 and 1.56. The full-order mixed H_2/H_∞ controller feedback gains obtained are K (second) = [–0.2864 –0.07848] and K (fourth) = [–0.9866 –0.7848 –0.6740 –0.0165]. With this feedback, the closed-loop system matrices formulated are:

$$A_c(second) = \begin{bmatrix} -1.8943 & 0.4540 \\ -10.2012 & -16.0754 \end{bmatrix} \quad \text{and} \quad A_c(fourth) = \begin{bmatrix} -1.1864 & -0.5848 & -0.6742 & -0.0165 \\ -0.986 & -1.2687 & -0.174 & -0.0146 \\ -0.9864 & -0.7848 & -0.6740 & 0.9835 \\ -0.9678 & -0.7848 & -1.674 & -2.0165 \end{bmatrix}$$

The robustness properties are studied by formulating an open-loop system $[K\,(sI - A)^{-1}B]$ using mixed H_2/H_∞ control law K derived for the above systems.

4.2 Formulation of H-infinity controllers

The H-infinity optimisation has become an effective design approach to tackle the problem of stability and performance robustness of a variety of plants. In this section, the H_∞ controller based on the mixed sensitivity approach is synthesised for the same second and fourth order plants described in equations (22)-(23). The optimisation problem is solved by selecting the weights to obtain better performance and robustness properties. The weight selection is similar to classical single loop shaping, namely specifying sufficient gain at low frequency for performance, good roll-off at high frequency to attenuate noise and attempting to achieve low enough roll-off rates at crossover to ensure good phase margin. The multi-variable case is slightly more complicated as each plant input may affect several outputs. This requires careful plant scaling and bandwidth selection. The open-loop transfer functions of the system formulated are:

$$G_2(s) = (16.15s + 56.35)/(s^2 + 18s + 30)$$
$$G_4(s) = (4s^3 + 8.8s^2 + 7.1s + 1.7)/(s^4 + 2.7s^3 + 2.5s^2 + 0.9s + 0.1)$$

Here, the objective is to find a stabilising controller K(s) for the plant G(s) to satisfy the following H_∞ inequality.

$$\left\| \begin{matrix} W_1(s)S(s) \\ W_3(s)T(s) \end{matrix} \right\|_\infty < 1$$

Where S(s) is the sensitivity function of the closed-loop system. The behaviour of this function with frequency indicates the affect of disturbance on the output of the system. The smaller the S(s) is, the less disturbance affect on the output. There are a variety of approaches both experimental [14] and numerical [15] discussed in the literature for the selection of weighting functions, so that the resulting controller can satisfy the desired design specifications. The weighting function W_1 (s) incorporated intentionally and is chosen to be large at those frequencies where the disturbance is more. The T(s) is the complementary sensitivity transfer function of the closed-loop system from the external input to the output. It plays an important role in characterising the robustness of the closed-loop system against plant perturbations. If T is smaller, then the system is more robust. The weighting function W_3 (s) is incorporated to shape the plant uncertainties. The smaller the $\|TW_3(s)\|_\infty$ is, the larger the perturbations that may be admitted without making the feedback system unstable.

To solve the mixed sensitivity problem, the plant G(s) is augmented with weighting functions W_1 (s) and W_3 (s) penalising the error and output signals, so that the closed-loop transfer function matrix is the weighted mixed sensitivity.

$$\text{i.e. } T_{zw} \stackrel{\Delta}{=} \begin{bmatrix} W_1 S \\ W_3 T \end{bmatrix}$$

Here the transfer functions G(s), W1(s) and W3(s) must be proper. Now, the objective is to solve the small gain infinity norm robust control problem, i.e. to find a stabilising controller K(s) for a system G(s) such that the closed-loop transfer function T_{zw} is minimised. This problem is solved for the constraint constant $\gamma = 1$. The plant G(s) is augmented with suitable loop shaping filters, so that the infinity norm concept provides direct shaping of closed-loop singular values. The robust mixed sensitivity based H-infinity controllers obtained for second and fourth order systems are:

$$K_{H\infty}(2nd\ order) = \frac{(s+16.45)(s+3.21)(s+2.45)}{(s+15.8836)(s+4.12)[s+(17.896\pm j0.492)]}$$

$$K_{H\infty}(4th\ order) = \frac{(s+2.84)[s+(0.899\pm j0.478)][s+(1.001\pm j0.001)][s+(0.996\pm j0.001)]}{(s+15.8836)(s+1.1)^2(s+0.4061)[s+(0.896\pm j0.492)][s+(1.0\pm j0.003)]}$$

This is done by using an optimisation procedure to provide internal stability, whilst simultaneously forcing the singular values of appropriate closed-loop operators like sensitivity function $S = (I + G(s)K(s))^{-1}$and the complementary sensitivity function $T = GK(I + G(s)K(s))^{-1}$ to lie below the specified bound over the desired frequency range. The weighting functions considered forces the 'S' to be small at low frequencies and DC, and also ensures that the reference signals are tracked adequately in the steady state. The high weighting on complementary sensitivity function T at high frequencies prevents the controller BW from becoming too high. The S(s) and T(s) obtained here are:

$$S(s) = \frac{(s+15.884)(s+1.125)(s+0.2)[s+(1.015\pm j0.029)][s+(0.984\pm j0.027)]}{(s+11.438)(s+0.5)^2(s+0.406)[s+(3.036\pm j2.349)][s+(0.896\pm j0.492)]}$$

$$T(s) = \frac{(s+2.839)(s+1.125)(s+0.407)^2(s+0.997)(s+1.003)[s+(0.899\pm j0.478)]}{(s+11.438)(s+0.996)(s+1.004)(s+0.406)^2[s+(0.896\pm j0.492)][s+(3.036\pm j2.349)]}$$

4.3 The Linear Quadratic Gaussian (LQG) Controllers

The LQG controller design methodology enables a controller to be synthesised which is optimal with respect to a specified quadratic performance index. Furthermore, this theory takes into account the presence of white noise disturbances acting on the system. In many control problems, it is straight forward to translate the required performance objective into a problem of minimising a quadratic cost functional. The LQG controller is formulated for the continuous time singularly perturbed system described in equation (23)-(24) using Robust Control System tool box in MATLAB software. The format of the system under consideration is:

$$\dot{x}(t) = Ax(t) + B_1 u(t) + B_2 w(t)$$

$$y(t) = C_1 x(t) + Du(t) + v(t)$$

Here, w(t) is the plant noise and v(t) is the measurement noise. These are sequences of independent gaussian vectors with zero mean values and covariances W and V, which minimise the quadratic cost and the covariance between the plant and measurement noise. Here the objective is to compute an optimal controller K(s) to stabilise the above described plant and minimise the following quadratic cost function:

$$J_{lqg} = \lim_{T\to\infty} E\{\int_0^T [x^T \quad u^T] \begin{bmatrix} Q & N_c \\ N'_c & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt\}$$

The plant noise and the measurement noise have the following correlation function:

$$E\{\begin{bmatrix} w(t) \\ v(t) \end{bmatrix} [w^T(t) \quad v^T(t)]\} = \begin{bmatrix} \xi & N_f \\ N_f & \Theta \end{bmatrix} \delta(t-\tau)$$

The final controller K(s) have the following form,

$$K(s) = \begin{bmatrix} A - K_f C - BK_c + K_f DK_c & K_f \\ K_c & 0 \end{bmatrix}$$

which can be realised as full state feedback and Kalman filter i.e.,

$$u = -K_c \hat{x}$$
$$\dot{\hat{x}} = A\hat{x} + Bu + K_f (y - C\hat{x} - Du)$$

The LQG controllers obtained for the second and fourth order systems are:

$$K_{lqg}(2nd\ order) = \frac{(s+12.34)}{(s+0.019)(s+56.78)}$$

$$K_{lqg}(4th\ order) = \frac{(s+0.421)[s+(1.11 \pm j0.299)]}{(s+51.80)(s+0.89)[s+(1.068 \pm j0.260)]}$$

The simulated time response for unit step input of the closed-loop mixed H_2/H_∞, LQG and H-infinity systems derived above are shown in the Figs. 2 and 3. The robustness properties such as the Gain and Phase margin of the open-loop transfer function of the above systems are shown in Table 1. The results shown enumerates that the mixed H_2/H_∞ has a better steady state response as compared to LQG and H-infinity and also the order of the controller is lower with respect to the H-infinity controllers derived.

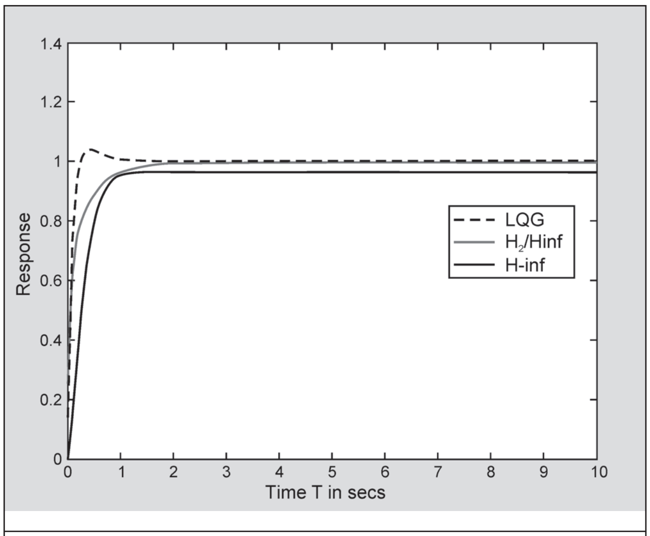

FIG. 2 TIME RESPONSE OF MIXED H_2/H-inf, H-inf AND LQG SYSTEM (2ND ORDER)

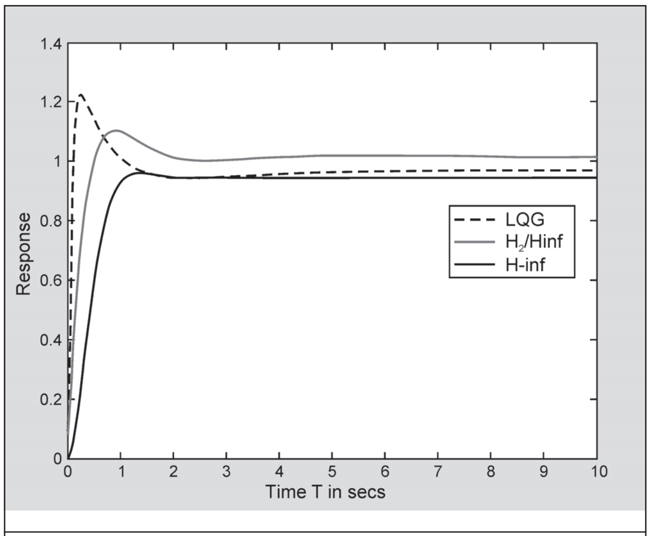

FIG. 3 TIME RESPONSE OF MIXED H_2/H-inf, H-inf AND LQG SYSTEM (4TH ORDER)

5.0 CONCLUSION

In this paper, a design procedure for a mixed H_2/H_∞ optimal state feedback controller for a singularly perturbed system has been proposed. During the formulation and simulation of the mixed H_2/H_∞ control law, an interesting feature observed was that, there exists a critical value of constraint constant below which the solution of mixed H_2/H_∞ controller could not be computed. The controllers obtained in this study are not unique and they depend largely on the choice of frequency dependent weighting functions and the quadratic weights. Usually the selection of these weighting functions is guided by the requirements of specific design application.

TABLE 1

ROBUSTNESS MEASURES

System	2nd order	4th order
Mixed H_2/H_∞	Gain Margin=infinity Phase Margin=infinity	Gain Margin=infinity Phase Margin=88.0 deg
LQG	Gain Margin=infinity Phase Margin=82.3 deg	Gain Margin=0.17 dB Phase Margin=61.0 deg
H-infinity	Gain Margin=infinity Phase Margin=95.1 deg	Gain Margin=infinity Phase Margin=62.7 deg

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